Localization improved gluon amplitudes in AdS

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Based on [2305.01016] with S. M. Chester and P. Ferrero
See also [2103.15830] with L. F. Alday, P. Ferrero and X. Zhou

Supersymmetric gauge theories

Schematic Lagrangian [Tachikawa; 1312.2684] :

$$\mathcal{L} = \frac{1}{2g_{YM}^2} tr[F_{\mu\nu} F^{\mu\nu}] + \frac{\theta}{16\pi^2} tr[F_{\mu\nu} \tilde{F}^{\mu\nu}] + \ldots$$

Supersymmetric gauge theories

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$$\tau(\Lambda) = \tau_{UV} - \frac{\beta_1}{2\pi i}\log\left(\frac{\Lambda}{\Lambda_{UV}}\right) + \sum_{k=1}^{\infty}\gamma_k\left(\frac{\Lambda}{\Lambda_{UV}}\right)^k$$

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$$\beta_1^{SU(N)} = 2N\left(1 - N_{adj}\right) - N_{\square} - (N-2)N_{\tiny\begin{array}{c}\square\\\square\end{array}}$$

$$\beta_1^{USp(2N)} = (2N+2)\left(1 - N_{adj}\right) - N_{\square} - (2N-2)N_{\tiny\begin{array}{c}\square\\\square\end{array}}$$

Common cases are:

- $N_{adj} = 1$ ($\mathcal{N} = 4$ SYM)
- $N_{\square} = 2N$ for $SU(N)$ or $N_{\square} = 2N + 2$ for $USp(2N)$
- $N_{\square} = 4, \; N_{\substack{\square \\ \square}} = 2$ for $SU(N)$ or $N_{\square} = 4, \; N_{\substack{\square \\ \square}} = 1$ for $USp(2N)$

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Plan: Focus on $USp(2N)$ SCFT at large N and study $\left\langle \mathcal{O}^A \mathcal{O}^B \mathcal{O}^C \mathcal{O}^D \right\rangle$ of $\Delta = 2$ Higgs branch operator $\mathcal{O}^A = \tilde{\phi} T^A \phi + c.c$ where T^A generates $SO(8)$ flavour symmetry.

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- Higgs branch $\mathcal{O} \in B\bar{B}[0;0]^{(R;0)}_{2R}$ means $Q^1_\alpha \mathcal{O} = \bar{Q}^2_{\dot{\alpha}} \mathcal{O} = 0$... for instance $\varphi^a \varphi^b$ where $\varphi = (\phi, \tilde{\phi})$

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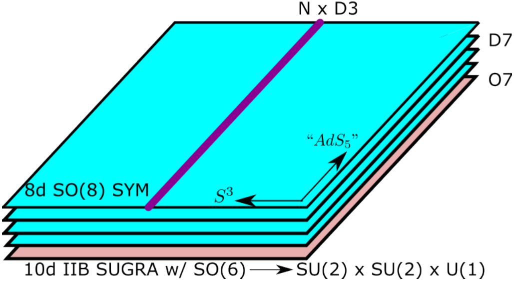
- Coulomb branch $\mathcal{O} \in B\bar{L}[0;0]^{(0;r)}_{r}$ means $Q^1_\alpha \mathcal{O} = Q^2_\alpha \mathcal{O} = 0$... for instance $tr(WW)$

Brane setup



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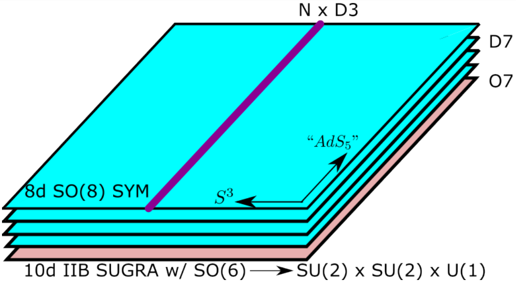

D3-D7 system leads to privileged $AdS_5 \times S^3$ [Aharony, Fayazuddin, Maldacena; 9806159] .

Single trace ops (KK modes) are from 10d with spin-2 and 8d with spin-1.

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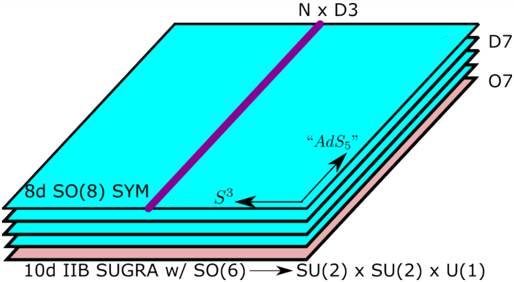

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$$A^A_\mu(x,y) = \sum_{\mathfrak{M}} A^A_{\mathfrak{M}}(x) Y^{\mathfrak{M}}_\mu(y) \Rightarrow (L,R) = \left(\tfrac{p-2}{2}, \tfrac{p}{2}\right) \oplus \left(\tfrac{p}{2}, \tfrac{p-2}{2}\right)$$

These operators in $B\bar{B}[0;0]_p^{(p/2;0)}$ include moment map as $p=2$.

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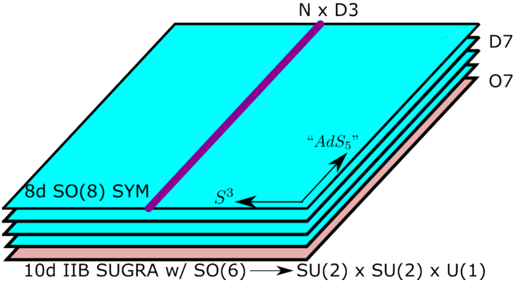

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$$\mathcal{O}^A(x, v) \equiv \mathcal{O}^A_{ab}(x) v^a v^b, \quad v_{ij} \equiv \epsilon_{ab} v^a_i v^b_j$$

CFT quantities involve $\alpha = \frac{v_{13} v_{24}}{v_{12} v_{34}}$ like $U = \frac{x^2_{12} x^2_{34}}{x^2_{13} x^2_{24}}$, $V = \frac{x^2_{14} x^2_{23}}{x^2_{13} x^2_{24}}$.

The key correlator

$G(U, V; \alpha)$ is a double expansion for large N and $\lambda = g_{YM}^2 N$.

$$G_{\mathcal{N}=4} = G_\emptyset + \frac{G_R}{N^2} + \ldots$$

Tree level	[Rastelli, Zhou; 1608.06624]

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$SU(N)$ 1-loop	[Alday, Bissi; 1706.02388] [Aprile, Drummond, Heslop, Paul; 1706.02822]
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Contact terms	[Chester; 1908.05247] [Chester, Pufu; 2003.08412]

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Finite τ upgrade	[Binder, Chester, Pufu, Wang; 1902.06263] [Chester, Green, Pufu, Wang, Wen; 1912.13365]
Virasoro-Shapiro	[Alday, Hansen, Silva; 2204.07542]

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$USp(2N)$ 1-loop	[Alday, Bissi, Zhou; 2110.09861] [Huang, Wang, Yuan, Zhou; 2301.13240]
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$$G^{ABCD}_{\mathcal{N}=2} = G^{ABCD}_{\emptyset} + \frac{G^{ABCD}_{F^2}}{N} + \frac{a^{ABCD}_{F^4}}{N^2} G_0 + \frac{G^{ABCD}_{R}}{N^2} + \frac{G^{ABCD}_{F^2|F^2}}{N^2} + \frac{\hat{a}^{ABCD}}{N^2} \log N G_0 + \dots$$

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Known terms

In terms of $U = z\bar{z}$ and $V = (1-z)(1-\bar{z})$, write [Nirschl, Osborn; 0407060] :

$$G^{ABCD}(z,\bar{z};\alpha) = \frac{z(1-\alpha\bar{z})f^{ABCD}(\bar{z}) - (z \leftrightarrow \bar{z})}{z-\bar{z}} + (1-\alpha z)(1-\alpha\bar{z})H^{ABCD}(z,\bar{z})$$

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Next use Mellin representation for $s+t+u=6$ [Mack; 0907.2407] :

$$H^{ABCD}(U,V) = \int \frac{dsdt}{(4\pi i)^2} U^{s/2}V^{t/2-2}M^{ABCD}(s,t)\Gamma[2-s/2]^2\Gamma[2-t/2]^2\Gamma[2-u/2]^2$$

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In terms of $\mathtt{c}_s = f^{ABE}f^{ECD}$, $\mathtt{d}_s = \delta^{AB}\delta^{CD}$ and $\mathtt{d}_{st} = f^{JAK}f^{KBL}f^{LCM}f^{MDJ}$:

$$M_{F^2} = \frac{\mathtt{c}_s}{(s-2)(u-2)} - \frac{\mathtt{c}_t}{(t-2)(u-2)}, \quad M_R = \frac{\mathtt{d}_s}{s-2} + \frac{\mathtt{d}_t}{t-2} + \frac{\mathtt{d}_u}{u-2}$$

$$M_{F^2|F^2} = \sum_{m,n=2}^{\infty} \frac{c_{mn}\mathtt{d}_{st}}{(s-2m)(t-2n)} + \frac{c_{mn}\mathtt{d}_{tu}}{(t-2m)(u-2n)} + \frac{c_{mn}\mathtt{d}_{us}}{(u-2m)(s-2n)} + a$$

$$c_{mn} = \frac{3m^2n + 3mn^2 - 4m^2 - 16mn - 4n^2 + 15m + 15n - 12}{(m+n-4)(m+n-3)(m+n-2)}$$

Fixing the rest

There are unknown constants (functions of λ) in

$$M_0^{ABCD} = b_1 \left[P_{\mathbf{1}}^{ABCD} + \frac{P_{\mathbf{35_v}}^{ABCD}}{5} + \frac{P_{\mathbf{300}}^{ABCD}}{15} \right] + b_2[P_{\mathbf{35_c}}^{ABCD} - P_{\mathbf{35_v}}^{ABCD}] + b_3[P_{\mathbf{35_s}}^{ABCD} - P_{\mathbf{35_v}}^{ABCD}]$$

$$\mathbf{28} \otimes \mathbf{28} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \oplus \mathbf{35_c} \oplus \mathbf{35_s} \oplus \mathbf{350} \oplus \mathbf{300}$$

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Superconformal twist

[Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees; 13]

Supersymmetric localization

[Pestun; 0712.2824]

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$$M_0^{ABCD} = b_1 \left[P_{\mathbf{1}}^{ABCD} + \frac{P_{\mathbf{35_v}}^{ABCD}}{5} + \frac{P_{\mathbf{300}}^{ABCD}}{15} \right] + b_2 [P_{\mathbf{35_c}}^{ABCD} - P_{\mathbf{35_v}}^{ABCD}] + b_3 [P_{\mathbf{35_s}}^{ABCD} - P_{\mathbf{35_v}}^{ABCD}]$$

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Superconformal twist

[Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees; 13]

$$\left[z \frac{\partial}{\partial z} - \frac{d-2}{2} \alpha \frac{\partial}{\partial \alpha} \right] G = 0$$

Exact OPE coefficients for Schur operators in $f(z)$ like $B\bar{B}[0;0]_{2R}^{(R;0)}$.

Supersymmetric localization

[Pestun; 0712.2824]

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↘ ↙

Same physics in 3D

[Chester, Lee, Pufu, Yacoby; 1412.0334] [Beem, Peelaers, Rastelli; 1601.05378]

The integrated constraint

For a deformed theory,

$$Z = \int \mathcal{D}X \ \exp\left[-S[X] + m_A \mathcal{O}^A\right] = \int \mathcal{D}X \ \exp\left[-S[X] + m_A \tilde{\phi} T^A \phi\right]$$

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Each "$Z^{-1}\partial_m^4 Z$" is an integral of $\left\langle \mathcal{O}^A \mathcal{O}^B \mathcal{O}^C \mathcal{O}^D \right\rangle$ [Chester; 2205.12978] .

$$\left. \frac{Z^{-1}\partial^4 Z}{\partial m_A \partial m_B \partial m_C \partial m_D} \right|_{m=0} = 16N^2 \sum_r P_r^{ABCD} I[H_r]$$

$$I[H] = \int \frac{dR d\theta}{\pi} R^3 \sin^2\theta H(U,V) \int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}-2} V^{\frac{t}{2}} \Gamma[-\tfrac{s}{2}]^2 \Gamma[-\tfrac{t}{2}]^2 \Gamma[-\tfrac{2+s+t}{2}]^2 \Big|_{z=Re^{i\theta}}$$

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After computing many digits [Caron-Huot, Coronado, Zahraee; ...] ,

$$I[1] = \frac{1}{24}, \quad I[s] = \frac{1}{12}, \quad I[\tfrac{1}{2-s}] = \frac{1}{16}, \quad I\left[\sum_{m,n} \frac{c_{mn}}{(s-2m)(t-2n)}\right] = \frac{11}{1296} - \frac{\zeta(3)}{108}$$

The localization matrix model

The object to differentiate is

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} dx_i x_i^2 \prod_{i<j} (x_i^2 - x_j^2)^2 \left| Z_{0-loop}(x, \tau_{UV}) Z_{1-loop}(x, m) Z_{inst}(x, \tau_{UV}, m) \right|^2$$

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Defining $m_{(2n)} = \frac{1}{4}\sum_{A=1}^{4} \mu_A^{2n}$,

$$Z = \int [dX] e^{-\frac{8\pi^2}{g_{YM}^2} tr(X^2)} \prod_{i=1}^{N} \frac{H(2x_i)^2}{\prod_A H(x_i \pm \mu_A)} = \int [dX] e^{-\frac{8\pi^2}{g_{YM}^2} tr(X^2) - S_{int}^{(0)} - m_{(2)} S_{int}^{(2)} - m_{(4)} S_{int}^{(4)} - \dots}$$

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Perturbations away from Gaussian model have nice expansions

$$S_{int}^{(0)} = 4 \sum_{j=1}^{\infty} \frac{(-1)^j}{j+1} \zeta(2j+1)(1 - 4^j) tr(X^{2j+2})$$

$$S_{int}^{(2n>0)} = 4 \sum_{j=1}^{\infty} \frac{(-1)^j}{j+1} \zeta(2j+1) \binom{2j+2}{2n} tr(X^{2j+2-2n})$$

Large N expansion

Since $S_{int} = 0$ gives massless $\mathcal{N} = 4$, cumulant expansion says

$$F = F_{\mathcal{N}=4} + \Delta F = F_{\mathcal{N}=4} + \langle S_{int} \rangle - \frac{1}{2!} \left\langle S_{int}^2 \right\rangle_c + \frac{1}{3!} \left\langle S_{int}^3 \right\rangle_c - \dots$$

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Using leading order $\left\langle tr(X^{2j}) \right\rangle$ as in [Beccaria, Dunne, Tseytlin; 2105.14729] ,

$$\Delta F = \frac{4N}{\sqrt{\pi}} \sum_{n=0}^{\infty} m_{(2n)} \sum_{j=1}^{\infty} \frac{(-1)^j}{j+1} \zeta[2j+1][1 - \delta_{n,0} 4^j] \binom{2j+2}{2n} \frac{\Gamma[j-n+\frac{3}{2}]}{\Gamma[j-n+3]} \left[\frac{\lambda}{4\pi^2} \right]^{j-n+1}$$

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Close the contour either way in

$$\Delta F = N \int \frac{ds}{2\pi i} \left(\frac{\lambda}{16\pi^2} \right)^s \frac{\Gamma(-s)}{\Gamma(s+2)} \left[4(4^s - 4)\Gamma(2s)\zeta(2s-1) \right.$$
$$\left. + 8 m_{(2)} \Gamma(2s+2)\zeta(2s+1) - \frac{2}{3} m_{(4)} \Gamma(2s+4)\zeta(2s+3) + O(\mu^6) \right] + O(1)$$

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Simple strong coupling expansion without the arcs

$$F = F_{\mathcal{N}=4} + N\left[\frac{\log 2}{2\pi^2}\lambda - \frac{1}{2}\log\lambda + \left(\log\pi + \frac{7}{3}\log 2 - 12\log A + \frac{3}{2}\right) - \frac{\pi^2}{2\lambda}\right.$$
$$\left. - \left(4\log\lambda + 4 + 8\gamma - 8\log(4\pi) + \frac{16\pi^2}{3\lambda}\right) m_{(2)} - \frac{16\pi^2}{3\lambda} m_{(4)} + O(\mu^6)\right] + O(1)$$

The Toda equation

Matrix models with single trace potential $-ytr(X^2) + V(X)$ obey

$$\log\left(-\partial_y^2 F\right) = 2F - F\big|_{N\mapsto N+1} - F\big|_{N\mapsto N-1}$$

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Expand with $y = 16\pi^2 N/\lambda$ to get [Beccaria, Korchemsky, Tseytlin; 2210.13871]

$$F = (N + \tfrac{3}{4} + 2m_{(2)})(N + \tfrac{1}{4} + 2m_{(2)})\log\left(\tfrac{16\pi^2 N}{\lambda} + 8\log 2\right) -$$
$$\log[G(N + \tfrac{5}{4} + 2m_{(2)})G(N + \tfrac{7}{4} + 2m_{(2)})] - \tfrac{1}{8}\log(16\pi) - 2m_{(2)}\log(8\pi) - 8m_{(2)}^2\log(4\pi)$$
$$+N[8\log G(\tfrac{3}{2}) - 2\log\pi - 8m_{(2)}(1+\gamma)] - (\tfrac{1}{32} + \tfrac{1}{3}m_{(2)} + \tfrac{1}{3}m_{(4)})\left(\tfrac{16\pi^2 N}{\lambda} + 8\log 2\right)$$

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Derivatives for constraining the correlator are now

$$-\partial_{\mu_1}^4 F|_{\mu=0} = \frac{32\pi^2}{\lambda}N + 6\log\lambda - 16\log 2 + 3f(N) + O(e^{-N}, e^{-\lambda})$$
$$-\partial_{\mu_1}^2\partial_{\mu_2}^2 F|_{\mu=0} = 2\log\lambda + f(N) + O(e^{-N}, e^{-\lambda})$$
$$-\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4} F|_{\mu=0} = O(e^{-N}, e^{-\lambda})$$

Reproducing the Veneziano amplitude

$$16N^2 I[\tfrac{1}{2}H_1 - \tfrac{1}{9}H_{35_v} + \tfrac{2}{9}H_{35_c} + \tfrac{2}{9}H_{35_s} + \tfrac{20}{21}H_{300}] = 8\log\lambda + 4f(N) + O(e^{-N}, e^{-\lambda})$$

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Tree level and first higher derivative correction are

$$M^{ABCD}(s,t) = -\frac{2}{N}\left[\frac{V(s,t,\lambda)}{(s-2)(t-2)} tr(T^A T^B T^C T^D) + \textit{crossed}\right]$$

$$V(s,t,\lambda) = 1 - \frac{24\zeta(2)}{\lambda}(s-2)(t-2) \qquad + \dots$$

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$$M(s,t) = \Gamma\left[\frac{1}{2}\sum_i \Delta_i - \frac{d}{2}\right]^{-1} \int_0^\infty d\beta\; \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} A(2\beta s, 2\beta t)$$

Flat limit [Penedones; 1011.1485] gives $\Gamma[1-\ell_s^2 s]\Gamma[1-\ell_s^2 t]/\Gamma[1-\ell_s^2(s+t)]$!

Reproducing the Veneziano amplitude

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$$M_1^{ABCD} = M_0^{ABCD} + b_4[\delta^{AB}\delta^{CD}s + \delta^{AD}\delta^{BC}t + \delta^{AC}\delta^{BD}u] + b_5[f^{ACE}f^{EBD}(t-2) + f^{ADE}f^{EBC}(u-2)]$$

Reproducing the Veneziano amplitude

$$16N^2 I[\tfrac{1}{2}H_1 - \tfrac{1}{9}H_{\mathbf{35_v}} + \tfrac{2}{9}H_{\mathbf{35_c}} + \tfrac{2}{9}H_{\mathbf{35_s}} + \tfrac{20}{21}H_{\mathbf{300}}] = 8\log\lambda + 4f(N) + O(e^{-N}, e^{-\lambda})$$

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Tree level and first higher derivative correction are

$$M^{ABCD}(s,t) = -\frac{2}{N}\left[\frac{V(s,t,\lambda)}{(s-2)(t-2)} tr(T^A T^B T^C T^D) + \textit{crossed}\right]$$

$$V(s,t,\lambda) = 1 - \frac{24\zeta(2)}{\lambda}(s-2)(t-2) + \frac{192\zeta(3)}{\lambda^{3/2}}(s-2)(t-2)(u-2) + \dots$$

$$M(s,t) = \Gamma\left[\frac{1}{2}\sum_i \Delta_i - \frac{d}{2}\right]^{-1} \int_0^\infty d\beta\, \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} A(2\beta s, 2\beta t)$$

Flat limit [Penedones; 1011.1485] gives $\Gamma[1-\ell_s^2 s]\Gamma[1-\ell_s^2 t]/\Gamma[1-\ell_s^2(s+t)]$!

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Which coupling to fix?

Since $\lambda = g_{YM}^2 N$ $\left(\textit{really } \frac{g_{YM}^2 N}{1+g_{YM}^2 \log 2/2\pi^2} \right)$, we can consider

Fixed λ:

- Keep residues
- Keep $O(e^{-\sqrt{\lambda}})$ arcs
- Neglect $O(e^{-N})$ instantons

Fixed τ:

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Theory has $SL(2,\mathbb{Z})$ duality group acting as [Seiberg, Witten; 9408099] :

$$S: \quad \tau \mapsto -1/\tau \quad \Leftrightarrow \quad \mathbf{35_v} \leftrightarrow \mathbf{35_c} \quad \Leftrightarrow \quad \mathcal{F}_\mathbf{v} \leftrightarrow \mathcal{F}_\mathbf{c}\,,$$
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Overview of the instanton calculation

Defining $q_{UV} = e^{2\pi i \tau_{UV}}$, [Nekrasov; 0206161] [Nekrasov, Shadchin; 0502180] found

$$Z_{inst}(x, \tau_{UV}, \mu) = \sum_{k=0}^{\infty} q_{UV}^k Z_k(x, \mu)$$

$$Z_k(x, \mu) = \int \prod_{I=1}^{\lfloor k/2 \rfloor} \frac{d\phi_I}{2\pi i} z_k(\phi, x) z_k^{\boxminus}(\phi, x) z_k^{\square}(\phi, \mu)$$

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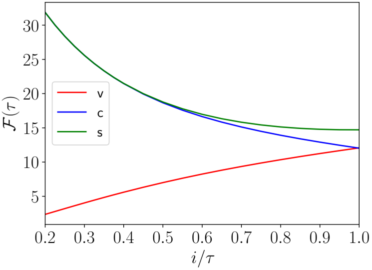
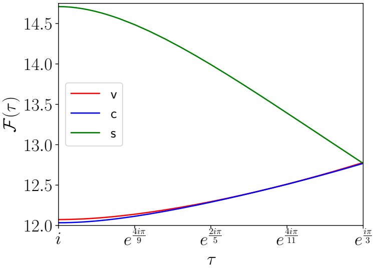
$$q_{UV}^2 = 16 \frac{\theta_2(\tau/2)^4}{\theta_3(\tau/2)^4} \quad \Leftrightarrow \quad \pi i \tau = \frac{\pi K(1 - q_{UV}^2/16)}{2K(q_{UV}^2/16)}$$

UV/IR relation in [Douglas, Lowe, Schwarz; 9612062] is reproduced.

Testing the extra contribution

Numerical localization inputs for $USp(4)$ to two instantons:





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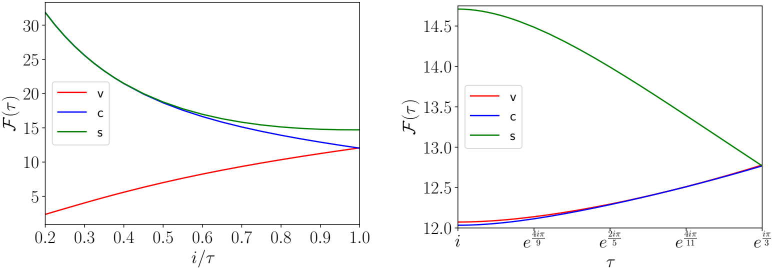

Function which matches modular ansatz up to two instantons:

$$Z_{extra} = 1 - 8\sqrt{q}\prod_{A=1}^{4}\mu_A + q\left[\frac{109}{8} - 3m_{(2)} - 2\sum_{A<B}\mu_A^2\mu_B^2 + 32\prod_{A=1}^{4}\mu_A^2\right]$$

Further directions

- Understanding how to incorporate more than two instantons is conceptually important and can help other approaches such as the numerical bootstrap [Chester, Dempsey, Pufu; 2111.07989] .
- Coulomb branch 3pt functions can be computed in the near future [Bissi, Fucito, Manenti, Morales, Savelli; 2112.11899] and perhaps 4pt functions (despite long multiplet exchange) later on.
- Strongly coupled Argyres-Douglas CFTs and perhaps their 4d $\mathcal{N} = 2$ S-folds [Apruzzi, Giacomelli, Schäfer-Nameki; 2001.00533] will be on the table if non-Lagrangian versions of these methods are worked out.

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