Problems Related to Finding Universal Gröbner Bases

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The need for multiple Gröbner bases

What are some monomial orders we might use?

- Homogenizing an ideal: graded orders.
- Eliminating variables: lexicographic orders.
- Tensor products of polynomial rings: product orders.

We wish to explore the possibility of using a Gröbner basis that works regardless of the monomial order. This is a *universal Gröbner basis* [Weipsfenning, 1987]. More generally we will be interested in how a Gröbner basis changes as we vary the monomial order.

Existence of universal Gröbner bases

Theorem

Let $\mathbb{K}$ be a field. Any ideal $I \subset S := \mathbb{K}[x_1, ..., x_n]$ has finitely many initial ideals $LT(I)$.

Idea

Assume Σ_0, the set of initial ideals of I is infinite. By the properties of leading terms, a descending chain can be constructed, $\Sigma_0 \supset \Sigma_1 \supset \Sigma_2 \supset ....$ Corresponding to this is an ascending chain of monomial ideals, $\langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \langle m_1, m_2, m_3 \rangle \subset ....$

If I is and ideal in S, we say two monomial orders $<_a$ and $<_b$ are equivalent if $LT_{<_a}(I) = LT_{<_b}(I)$. There are finitely many equivalence classes.

Existence of universal Gröbner bases

Corollary

Every ideal $I \subset S$ posesses a finite, universal Gröbner basis.

Proof.

For the monomial orders on I, choose one element from each equivalence class. For each representative order, find the Gröbner basis. By construction this is a Gröbner basis for any equivalent order. Take the union of all such Gröbner bases. □

How may one find an ideal's universal Gröbner basis in practice? At each stage of Buchberger's algorithm, consider every way that a monomial order could possibly distinguish between your monomials. Run Buchberger's algorithm that many times.

An Example

Consider the ideal defining the affine twisted cubic: $I = \langle y - x^2, z - x^3 \rangle \subset \mathbb{Q}[x, y, z]$. Running the algorithm, we can make the following chart:

Initial Ideal	Gröbner Basis
$\langle y^3, xz, xy, x^2 \rangle$	$\{y^3 - z^2, xz - y^2, xy - z, x^2 - y\}$
$\langle z^2, xz, xy, x^2 \rangle$	$\{y^3 - z^2, xz - y^2, xy - z, x^2 - y\}$
$\langle y^2, xy, x^2 \rangle$	$\{y^2 - xz, xy - z, x^2 - y\}$
$\langle y, x^3 \rangle$	$\{x^2 - y, x^3 - z\}$
$\langle y, z \rangle$	$\{x^2 - y, x^3 - z\}$
$\langle z, x^2 \rangle$	$\{z - xy, x^2 - y\}$

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Initial Ideal	Gröbner Basis
$\langle y^3, xz, xy, x^2 \rangle$	$\{\underline{y^3} - z^2, \underline{xz} - y^2, \underline{xy} - z, \underline{x^2} - y\}$
$\langle z^2, xz, xy, x^2 \rangle$	$\{y^3 - \underline{z^2}, \underline{xz} - y^2, \underline{xy} - z, \underline{x^2} - y\}$
$\langle y^2, xy, x^2 \rangle$	$\{\underline{y^2} - xz, \underline{xy} - z, \underline{x^2} - y\}$
$\langle y, x^3 \rangle$	$\{x^2 - \underline{y}, \underline{x^3} - z\}$
$\langle y, z \rangle$	$\{x^2 - \underline{y}, x^3 - \underline{z}\}$
$\langle z, x^2 \rangle$	$\{\underline{z} - xy, \underline{x^2} - y\}$

Varrying term orders

For a non-negative integer matrix,

$$M = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{bmatrix}$$

let $x^\alpha >_M x^\beta$ if $\alpha \cdot \mathbf{w}_1 > \beta \cdot \mathbf{w}_1$ or if $\alpha \cdot \mathbf{w}_1 = \beta \cdot \mathbf{w}_1$ and $\alpha \cdot \mathbf{w}_2 > \beta \cdot \mathbf{w}_2$ and so on.

Theorem

A relation on $S := \mathbb{K}[x_1, ..., x_n]$ is a monomial order if and only if it is $>_M$ for some matrix M [Robbiano, 1985].

Varrying term orders

Suppose $\mathcal{G} = \{g_1, ..., g_t\}$ generates an ideal whose initial ideal is $\left\langle x^{\alpha(1)}, ..., x^{\alpha(t)} \right\rangle$. We want this to be equal to $\langle LT(g_1), ..., LT(g_t) \rangle$. This will be true with respect to a monomial order whose first weight vector is in:

$$C_{\mathcal{G}} = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} : (\alpha(i) - \beta) \cdot \mathbf{w} > 0 \text{ whenever } x^\beta \text{ appears in } g_i\}$$

These are open cones in the positive orthant of $\mathbb{R}^n$.

Varrying term orders

Theorem

Let $I \subset S$ be an ideal. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are distinct marked Gröbner bases of I, $C_{\mathcal{G}_1} \cap C_{\mathcal{G}_2} = \emptyset$. The union of $\overline{C_{\mathcal{G}}}$ over all marked Gröbner bases $\mathcal{G}$ is the entire positive orthant without the origin.

Proof.

If $\mathbf{w}$ belongs to the intersection then $\mathcal{G}_1$ and $\mathcal{G}_2$ are Gröbner bases for I with respect to some monomial order that has $\mathbf{w}$ as its first weight. I has only one initial ideal for this fixed order but $\mathcal{G}_1$ and $\mathcal{G}_2$ were chosen to give it different initial ideals. To prove that the union of the closures is $\mathbb{R}^n_{\geq 0} \backslash \{0\}$, note that every such vector is the first weight vector of *some* monomial order. $\square$

The Gröbner Fan

The collection of these cones is the *Gröbner fan* of an ideal [Mora & Robbiano, 1988]. It divides the positive orthant into chambers.

Example

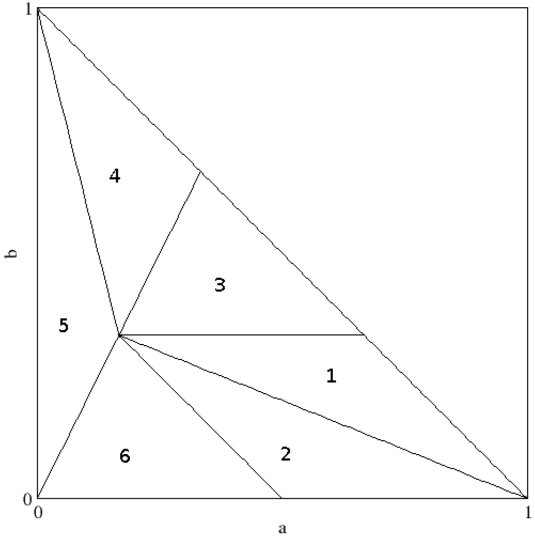
Recall that one of the marked Gröbner bases for the twisted cubic was $\mathcal{G}_3 = \{\underline{y^2} - xz, \underline{xy} - z, \underline{x^2} - y\}$. **w** is in $C_{\mathcal{G}_3}$ if and only if, it corresponds to an order that causes the leading terms of $\mathcal{G}_3$ to be the ones underlined. The conditions are:

- $(0, 2, 0) \cdot (a, b, c) > (1, 0, 1) \cdot (a, b, c)$
- $(1, 1, 0) \cdot (a, b, c) > (0, 0, 1) \cdot (a, b, c)$
- $(2, 0, 0) \cdot (a, b, c) > (0, 1, 0) \cdot (a, b, c)$

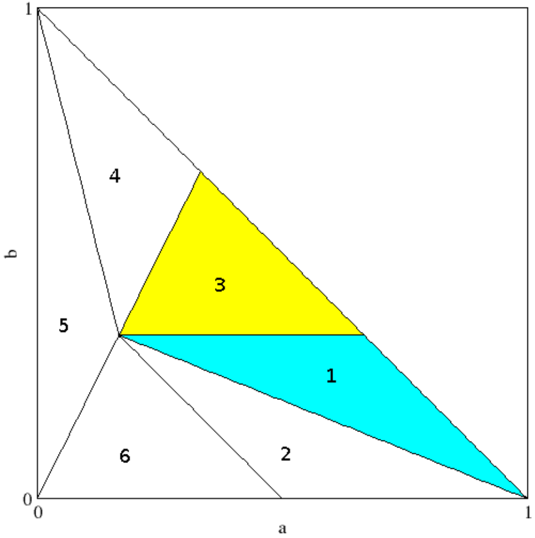
These define the cone:

$$C_{\mathcal{G}_3} = \{(a, b, c) \in \mathbb{R}^n_{\geq 0} : 2b > a + c, a + b > c, 2a > b\}$$

The Gröbner Fan



The Gröbner Fan



Example

Consider $f_1, ..., fn \in \mathbb{K}[t_1, ..., t_m]$. If $J = \langle x_1 - f_1(t_1, ..., t_m), ..., x_n - f_n(t_1, ..., t_m)\rangle$, polynomial implicitization tells us to compute $I := J \cap \mathbb{K}[x_1, ..., x_n]$. We do not need to solve for a Gröbner basis for the t elimination order. We already have a Gröbner basis for the x elimination order.

Algorithm

Given a Gröbner basis for a starting order, the Gröbner walk can convert it to the Gröbner basis for a target order quickly by following a path from cone to cone and performing a computation at each boundary [Collart, Kalkbrener & Mall, 1997].